# Research statement

Riccardo Pedrotti

My main area of research is **symplectic geometry** with **an interest in low dimensional topology**. More precisely I am using tools from Floer theory and pseudo-holomorphic techniques to study symplectic 4-manifolds. The reason why I find this approach worth pursuing is that it can lead to a more "concrete" description of several invariants of a given manifold, making them easier to compute in many cases. Following this philosophy, together with Timothy Perutz, **I am working towards a program that aims to describe the Seiberg-Witten invariants of a Lefschetz fibration over the sphere in terms of the geometry of the regular fiber, a surface, and the thimbles.** As a necessary step towards the end goal, I generalised an identification of the Floer homology of a surface twist by P. Seidel [Sei96] to higher on higher dimensional manifolds [Ped24].
Central tools in my work are **Seidel's exact triangles** for Lagrangian ([Sei03])

$$\begin{tikzcd} HF^*(L_0, L_1) \arrow[r, "\sigma^{\mathrm{Lag}}"] & HF^*(L_0, \tau_V L_1) \arrow[r] & HF^*(L_0, V) \otimes HF^*(V, L_1) \arrow[ll, bend left, "\mu_2"] \end{tikzcd} \tag{1}$$

and Fixed Point Floer homology ([Sei01],

$$\begin{tikzcd} HF_{-*}(\phi) \arrow[r, "\sigma^{FP}"] & HF_{-*}(\tau_V \phi) \arrow[r] & HF^*(\phi V, V) \arrow[ll, bend left, "OC"] \end{tikzcd} \tag{2}$$

whose maps will be explained later. While Lagrangian Floer (co)homology is widely known, due to its involvement in the definition of the Fukaya category of a symplectic manifold $M$, Fixed Point Floer homology (FP) is a lesser known generalization of Hamiltonian Floer homology. It was introduced by Dostoglou and Salamon [DS94] as a way to count flat connections of the mapping torus of an automorphism of a nontrivial $SO(3)$-bundle over a surface $\Sigma$.
For a symplectomorphism $\phi : M \to M$ (with non-degenerate fixed points) of a symplectic manifold $(M, \omega)$, Dostoglou and Salamon defined $HF_*(\phi)$ as the Morse homology of a twisted loop space of $M$. The Hamiltonian version can be retrieved by setting $\phi = \mathrm{Id}$ in which case we compute the Morse cohomology of the (untwisted) loop space of $M$ (or a cover thereof) giving us the classical construction for $HF^*(M)$. For the purpose of my work, it is better to use a more geometrical definition of FP Floer homology, introduced by Seidel in his PhD thesis [Sei97]. Given a symplectomorphism $\phi : (M, \omega) \to (M, \omega)$ as above, generators of $CF_*(\phi)$ are given by *horizontal* sections of the mapping torus $\pi : M_\phi \to S^1$ (*horizontal* with respect to the distribution obtained from the 2-form $\omega_\phi \in \Omega^2(M_\phi)$, induced by $\omega$). On the other hand, the differential is given by counting index 1 finite energy pseudo-holomorphic sections of the bundle

$$\mathrm{Id} \times \pi : \mathbb{R} \times M_\phi \to \mathbb{R} \times S^1$$

"connecting" two given horizontal sections of $M_\phi$. This change of perspective makes it easier to define non-trivial maps between FP Floer homology groups, by counts of pseudo-holomorphic sections of certain maps called **Lefschetz fibrations**: roughly speaking, proper maps $\pi : E^{2n+2} \to \Sigma$ such that around each critical points of $\pi$ we can find holomorphic coordinates $(z_1, \dots, z_n)$ satisfying

$$\pi(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2.$$

The maps $\sigma^{\mathrm{Lag}}, \sigma^{FP}$ are an example of maps constructed in this way.

## 1. My research program

My research work broadly aims to address the following situation: given a sequence of positive twists

$$\phi = \tau_{V_k} \circ \dots \circ \tau_{V_1} \in \mathrm{MCG}(\Sigma)$$

in the mapping class group of a surface $\Sigma$, there is a procedure to build a 4-dimensional Lefschetz fibration over the (closed) disk $D$ from it: the regular fiber is $\Sigma$, in the interior of $D$ we have $k$ critical points and

the monodromy around $\partial D$ is equal to $\phi$. If $\phi$ happens to be a trivial word, i.e. isotopic to the identity, we can then "cap off" the disk to obtain a Lefschetz fibration over the 2-sphere $\pi : X \to S^2$ by gluing a trivial fibration over another disk and identifying them along their boundaries. If the genus of $\Sigma$ is at least 2, this capping off procedure is essentially unique.
The total space $X$ is a closed symplectic 4-manifold ([Gom04]), which then comes with a concrete description: the general fiber $\Sigma$ and the positive factorization of the identity $\mathrm{Id} = \tau_{V_1} \circ \cdots \circ \tau_{V_k} \in \mathrm{MCG}(\Sigma)$ used to define the fibration over the disk. **It is only natural to ask if we can read off the invariants of $X$ from such presentation.**
There are a couple of positive answers to this question. The Euler characteristic, and more generally the homology of $X$, is easily retrieved; I. Smith [Smi99] and B. Ozbagci [Ozb02] were able to compute the signature of $X$ using its description as a total space of a Lefschetz fibration. **I am currently working on computing an explicit formula for the Seiberg-Witten invariants** of $(X, \omega)$ following this *philosophy*. These invariants have a known interpretation in symplectic geometry, [Tau99][DS03] and [Ush04], which we are implicitly using. They can be thought of as a map

$$SW_X : \mathrm{Spin}^c \cong H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

from the set of $\mathrm{Spin}^c$-structures of $X$ to the integers, given by counting solutions to certain equations. Taubes proved, in his seminal work [Tau99], that these invariants are equal to the Gromov-Witten invariants $Gr$

$$Gr : H^2(X;\mathbb{Z}) \to \mathbb{Z} \tag{3}$$

given by assigning to each class $\alpha \in H^2(X;\mathbb{Z})$ a certain weighted count of compact, pseudo-holomorphic maps $u : \Sigma \to X$, whose fundamental class $u_*[\Sigma]$ is equal to the Poincare' dual of $\alpha$.
**This deep result provides the bridge from $SW_X$ to something that is amenable to Floer theory, i.e. counting of pseudo-holomorphic curves in $X$.**
In our setting, given a 4-dimensional Lefschetz fibration $X \to S^2$, by removing a little disk with no critical values from the base we obtain a fibration $\pi : X_{|D} \to D$ with trivial monodromy. We then attach a suitable cylindrical end to $D$ and extending the fibration over it, to get a (homotopy equivalent) fibration $\pi : X' \to \mathbb{C}$ suitable for Floer's techniques. **We then focus on counting finite energy pseudo-holomorphic sections $s$ while keeping track of their relative homology class $s_*[D, \partial] \in H_2(X, \partial X)$, to get an explicit formula for the map**

$$Gr : H^2(X;\mathbb{Z})_{sec} \to \mathbb{Z}$$

**restricted to the duals to homology classes represented by sections.**
To obtain the formula we observed that by stretching $\mathbb{C}$ along concentric circles, we can decompose our fibration $\pi$ into a sequence of Lefschetz fibrations $\pi_1, \ldots, \pi_k$ over annuli $A_1, \ldots, A_k$ equipped with cylindrical ends (the last one is a degenerate annulus with one end "pinched", hence a disk) and whose fibrations over them only have one critical point. Each pseudo-holomorphic section $s$ then induces a sequence of pseudo-holomorphic sections $s_1, \ldots, s_k$ with pairwise matching limits over the ends.

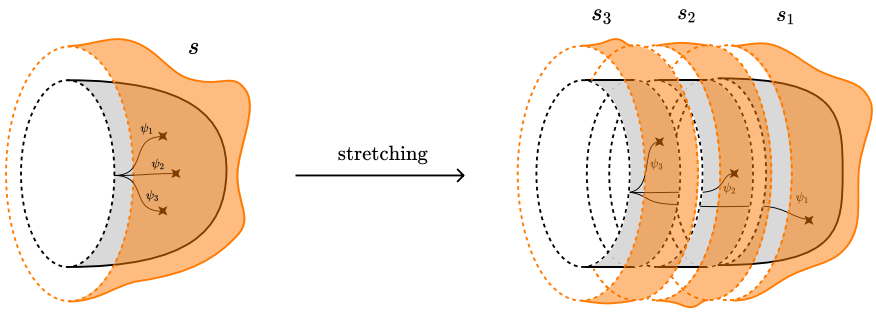



**Figure 1:** A Lefschetz fibration over the disk $D$ (colored in grey) with 3 critical values and a section $s$ (in orange), is decomposed into 3 Lefschetz fibrations over annuli with corresponding sections $s_1, s_2$ and $s_3$.

A gluing result then suggests that for "long enough" ends, **there is a bijection between pseudo-hol. sections of $\pi : X' \to \mathbb{C}$ and sequences of pseudo-hol. sections over such annuli.** "Detecting" the latter can be done as follows: each Lefschetz fibration $\pi_i : X_i \to A_i$ corresponds to the fibration defining the map

$$\sigma_i^{FP} : CF(\tau_{V_{i-1}} \cdots \tau_{V_1}) \to CF(\tau_{V_i} \tau_{V_{i-1}} \cdots \tau_{V_1})$$

in (2). Hence, understanding such map algebraically means being able to count the sections $s_i$'s. To keep track of the (relative) homology class of $s$, we exploit the topology of $X_{|D}$: its second homology group is

generated by the fundamental class of the regular fiber and the relative classes of the thimbles $\Delta_i$ over $\psi_i$. If we want to recover the homology class of $s$, Fig. 1 suggests that $s_1$ carries information about $\Delta_1$, while $s_3$ has to remember the intersection number with $\Delta_3$ and with the "tails" of the thimbles $\Delta_1$ and $\Delta_2$, since their vanishing paths traverse $A_3$. **To do so, we first constructed a local coefficients system $\mathcal{L}$ for** (2) **which allows an (enriched) version of the map $\sigma_i^{FP}$ to keep track of these intersection numbers.** We then use the chain homotopy between $CF(\tau_V\phi;\mathcal{L})$ and $\mathrm{Cone}(OC;\mathcal{L})$, provided by the triangle itself, to obtain **a description of the "enriched" $\sigma_i^{FP}$ in terms that are easier to compute.** Concretely, this is obtained by means of the mapping cone recognition lemma (MCR) [OS05, Lemma 4.4], together with an explicit description of the maps in the exact triangle.

### 1.1 A geometric proof of the exactness of (1),(2)

To be able to apply the MCR lemma in order to prove exactness of the triangles with $\mathbb{Z}_2$ coefficients we first described all the maps explicitly. In the Lagrangian case, as conjectured by P. Seidel, we are able to show that the map $\lambda$ is indeed the coproduct map obtained by counting sections of the trivial fibration over the thrice-punctured disk. **The proof that $\mu_2 \circ \lambda \sim 0$ is based on a novel ideal not present in the literature so far**. The composition $\mu_2 \circ \lambda$ counts index 0 sections over two thrice-punctured disks glued along two of these boundary punctures. By seeing this configuration as one end of the moduli space of annuli with varying conformal parameter and appropriate Lagrangian boundary conditions, we defined a chain-homotopic map by means of counting the sections over the nodal configuration of the other end, obtained as two disks, one of them carrying two boundary marked points, joined along an interior cylindrical end. If we momentarily focus on the fibration over the disk with 3 marked points, it defines a relative invariant of the mixed kind, (see [Sei01, page 14]) $\eta \in HF^*(L_0,L_1)\otimes HF^*(L_0,\tau_V L_1)\otimes HF^*(\tau_V^{-1})$. **Therefore we can conclude $\mu_2 \circ \lambda \sim \eta(-,c)$ for a certain cocycle $c \in CF^*(\tau_V^{-1})$, obtained by counting rigid pseudo-holomorphic sections of the fibration over the disk with cylindrical end at the origin.** The null-homotopy is then a consequence of the following result

**Theorem 1.** *([Ped24, Theorem 1.4]) Let $(\Sigma,\omega)$ be a orientable surface with genus at least 2, let $V$ be a framed essential Lagrangian curve in it, then*

$$[c] = 0 \in HF^*(\tau_V^{\pm 1})$$

The techniques used to prove Theorem 1 can actually be used to prove much more than what strictly needed, leading to this generalisation of a classic result of Seidel [Sei96] about twists on surfaces to a much broader class of symplectic manifolds, for example Calabi-Yau and Fano manifolds:

**Theorem 2.** *([Ped24, Theorem 1.1]) Let $(M,\omega)$ be a closed $w^+$-monotone symplectic manifold of dimension $2n \geq 4$ such that the symplectic class $[\omega]$ admits a rational representative. Let $V_1,\dots,V_m$ be pairwise disjoint framed Lagrangian spheres and set $\tau := \tau_{V_1}^{\sigma_1}\cdots\tau_{V_m}^{\sigma_m}$, where $\sigma_i = \pm 1$ for all $i$. Let $C_+$ (resp. $C_-$) be the union of all $V_i$'s such that $\sigma_i = +1$ (resp. $\sigma_i = -1$), then*

$$HF^k(\tau;\Lambda_\omega) \cong \bigoplus_{j=k \pmod 2} H^j_{Morse}(M \setminus C_-, C_+;\Lambda_\omega)$$

*where the Floer cohomology of $\tau$ is $\mathbb{Z}_2$-graded and $\Lambda_\omega$ is the Novikov field associated to the symplectomorphism $\tau : M \to M$.*

The proof of Theorem 2 (and with some minor modifications, of Thm 1) is based on these three main ideas:

1. By carefully choosing the Hamiltonian perturbation, the cochain groups $CF^*(\tau_V)$ and $C^*_{\mathrm{Morse}}(M,V)$ coincides, "modulo" certain *bad* pseudo-holomorphic strips that pass through $V$.
2. **The more the tubular neighbourhood around $S(T^*V)$ is stretched, the more energy a *bad* trajectory must have.**
3. Thanks to a rather delicate energy filtration argument, originally due to K. Ono [Ono95] in the Hamiltonian case, one can show that *bad* trajectories do not count towards the final result.

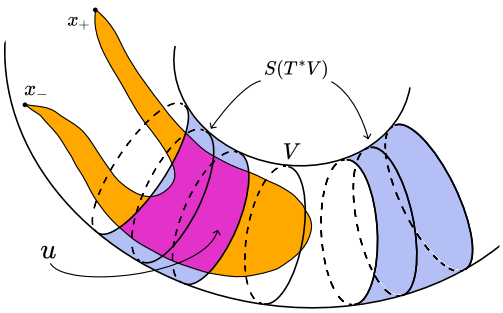



**Figure 2:** A curve $u$ going through $V$. The purple area in the curve can be proved to be arbitrarily big for long necks.

We conclude by remarking that Ghiggini and Spano [GS22] independently gave a rather explicit proof of the exactness of (2).

### 1.2 Constructing the local coefficient system and results in the Lagrangian case

To show how the local coefficients are built and work, we focus on the Lagrangian exact triangle (1). Fig. (1), which motivates the program, has its Lagrangian counterpart in Fig. (3)

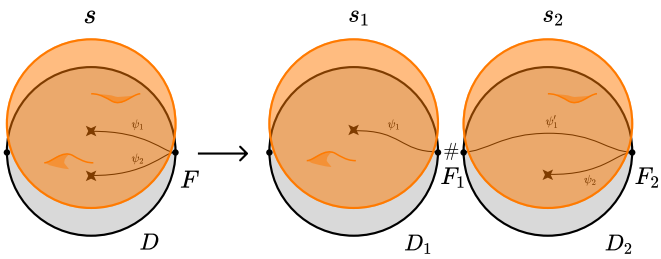



**Figure 3**

We count sections of a Lefschetz fibration $\pi : X \to D$ over the (closed) disk with appropriate Lagrangian boundary conditions by decomposing $X$ as a fiber sum of $X_i \to D_i$'s, each of them containing a single critical value. As before, each section $s_i$ contributes to the appropriate map $\sigma$ in the Lagrangian version of the exact triangle (2).

The local coefficient system for the Lagrangian exact triangle can be build as follows: choose a finite family $C$ of curves in $F \cong \Sigma$. Let $\ker(\text{incl.}_*) \subseteq \mathbb{Z}\langle C\rangle \xrightarrow{\text{incl.}_*} H_1(F)$, i.e. all the "combinations" of the curves in $C$ that are nullhomologous in $F$. Given a point $x$ on the surface, we set

$$\mathcal{L}_{Cx} := \mathbb{Z}\big[\hom_{\mathbb{Z}}\big(\ker(\text{incl.}_*), \mathbb{Z}\big)\big]$$

where $\mathbb{Z}[G]$ denotes the group ring of the group $G$. In practice, one has to use a little algebraic "trick" in order to define the the local coefficient system for points lying on the curves in $C$. Given a Lefschetz fibration $\pi : X \to D$ with regular fiber $F$ over a point in $\partial D$ and critical values $c_1, \dots, c_k$, **there is a canonical family of curves we can use to construct our local coefficient system on** $F$, i.e. by taking the family $\{V_j\}^k$ of vanishing cycles in the reference fiber $F$. In Fig. (3), $F_1$ is naturally equipped with $\mathcal{L}_{\{V_1\}}$ while $F_2$ comes together with $\mathcal{L}_{\{V_1,V_2\}}$.

Briefly, the action of the Floer differential with these local coefficients is given by adding the contribution of the homomorphism induced by *taking the intersection number with* $\gamma$. In the situation of Fig. (4),

$$\partial^{Floer}(x) = y \otimes T^{[\gamma \frown -]}$$

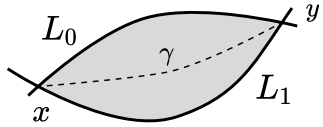



**Figure 4**

where $T$ is the formal variable in the group ring. Following what suggested by Fig. (3), by setting $L_1^{i-1} := \tau_{V_{i-1}} \cdots \tau_{V_1} L_1$, **each section** $s$ **and** $s_i$ **naturally contributes to the respective chain maps with local coefficients:**

$$\begin{aligned} &\widetilde{\sigma_{tot}^{Lag}} : CF(L_0, L_1) \to CF(L_0, \tau_{V_k} L_1^{k-1}; \mathcal{L}_{\{V_j\}^k}) \\ &\widetilde{\sigma_{V_i}^{Lag}} : CF(L_0, L_1^{i-1}; \mathcal{L}_{\{V_j\}_1^{i-1}}) \to CF(L_0, \tau_{V_i} L_1^{i-1}; \mathcal{L}_{\{V_j\}^i}). \end{aligned} \tag{4}$$

The fact that the sections $s, s_i$ "change" the local coefficient system in the target is a consequence of the fact that over $D_1$, $s_1$ does not carry any intersection number on the incoming end and tracks the intersection with $\Delta_1$, while $s_2$ "carries" the information about the intersection with the tail of $\Delta_1$ and with the new thimble $\Delta_2$. This is encoded algebraically by a map $\mathcal{A}$, which we called the "addition" map. $\mathcal{A}$ "adds" an element of $\mathcal{L}_C$ with the contribution of a pseudo-holomorphic section of a Lefschetz fibration with vanishing cycles $\{V_j\}$ over a regular fiber to get an element of $\mathcal{L}_{C \cup \{V_j\}}$.

**Theorem 3** ([PP]). *Let $L_0, L_1$ and $V \in \Sigma$ be essential Lagrangians. Let $\mathcal{L}_C$ be a local coefficient system as defined above, then the following triangle is exact:*

$$\begin{tikzcd} HF^*(L_0, L_1; \mathcal{L}_C) \arrow[r, "\widetilde{\sigma^{Lag}}"] & HF^*(L_0, \tau_V L_1; \mathcal{L}_C) \arrow[r, "\widetilde{\lambda}"] & HF^*(L_0, V; \mathcal{L}_C) \otimes HF^*(V, L_1) \arrow[ll, bend left, "\widetilde{\mu_2}"] \end{tikzcd} . \tag{5}$$

*The map $\widetilde{\lambda}$ is essentially the coproduct map, together with an identification induced by $\tau_V^{-1}$. $\widetilde{\lambda}$ and $\widetilde{\mu_2}$ act on the local coefficient system in a analogous way as $\partial^{Floer}$, by keeping track of intersection numbers with appropriate paths. The map $\widetilde{\sigma^{Lag}}$ is obtained from the map $\widetilde{\sigma_V^{Lag}}$ in local coefficient by forgetting the contribution of the intersection number with the thimble associated to $V$.*

For example, in Fig. (3), the map $\widetilde{\sigma^{Lag}}$ will record the section $s_2$ together with the intersection number with the thimble $\Delta_1$ extended along $\psi_1'$, but not the intersection number of the thimble over $\psi_2$.

**The map $\widetilde{\sigma_V^{Lag}}$ from** (4) **does not fit into the enriched triangle** (5). The reason being that keeping

track of the intersection number with the thimble $(\Delta, V)$ will make the compositions $\widetilde{\lambda} \circ \widetilde{\sigma_V^{Lag}}$ or $\widetilde{\sigma_V^{Lag}} \circ \widetilde{\mu_2}$ non-zero in (co)homology. This is not just a technicality but rather a consequence of the vanishing result from [Sei03, Prop 2.13], which is crucial in proving the exactness of the triangle. In dimension 2, the result is based on the fact that sections with opposite intersection number with $V$ will "cancel" each other.

**To retrieve the information about the "full" map $\widetilde{\sigma_{V_i}^{Lag}}$ in (4), we use the mapping cone recognition lemma to obtain an explicit identification**

$$CF\big(L_0, \tau_{V_i} L_1^{i-1}; \mathcal{L}_{\{V_j\}^i}\big) \simeq CF\big(\mathrm{Cone}\big(\widetilde{\mu_2}; \mathcal{L}_{\{V_j\}^i}\big)\big)$$

which can be used to prove our main result. Let $V_i^*$ be the homomorphism on $\ker(\mathrm{Incl}._*)$ induced by projecting on the $V_i$-factor.

**Theorem 4** ([PP]). *In the setting of the previous theorem, the map*

$$\widetilde{\sigma_{V_i}^{Lag}} : CF(L_0, L_1^{i-1}; \mathcal{L}_{\{V_j\}^{i-1}}) \to CF(L_0, \tau_{V_i} L_1^{i-1}; \mathcal{L}_{\{V_j\}^i}) \xrightarrow{\simeq} CF\big(Cone\big(\widetilde{\mu_2}; \mathcal{L}_{\{V_j\}^i}\big)\big)$$

*is chain-homotopic to the following map*

$$\begin{aligned} CF\big(L_0, L_1^{i-1}; \mathcal{L}_{\{V_j\}^{i-1}}\big) &\to CF\big(Cone\big(\widetilde{\mu_2}; \mathcal{L}_{\{V_j\}^i}\big)\big) \\ x \otimes T^{[f]} &\mapsto \big(\widetilde{\lambda}_{w_i}, \widetilde{Id}_{w_i} + \widetilde{\Psi}^{\heartsuit}_{w_i}\big) \end{aligned}$$

*where the maps are explicitly given. Here,*

$$\widetilde{\lambda}_{w_i}\big(x \otimes T^{[f]}\big) = \sum_{u \in \mathcal{M}_0^{\triangle}([u])} \sum_{\substack{y \in L_0 \cap V_i \\ z \in V_i \cap L_1^{i-1}}} |\mathcal{M}_0^{\triangle}([u])| \cdot y \otimes \Big(\mathcal{A}([f], u) \cdot (T + T^{-1})^{V_i^*}\Big) \otimes z$$

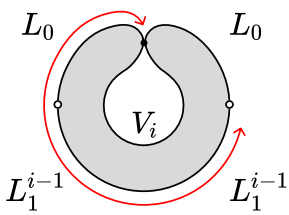
*where $\mathcal{M}^{\triangle}([u])$ is the 0-dimensional component of the moduli space of (pseudo-holomorphic) triangles in $\Sigma$ with sides lying on $L_0, L_1^{i-1}$ and $V_i$ and prescribed asymptotic conditions.*
*The second map acts as follows:*

$$\widetilde{Id}_{w_i}\big(x \otimes T^{[f]}\big) = x \otimes \mathcal{A}([f], 0)$$

*namely, it is the "inclusion" of local coefficients $\mathcal{L}_C \to \mathcal{L}_{C \cup \{V_i\}}$, and finally the last map:*

$$\widetilde{\Psi}^{\heartsuit}_{w_i}\big(x \otimes T^{[f]}\big) = \sum_{\substack{w \in L_0 \cap L_1^{i-1} \\ u' \in \mathcal{M}^{\heartsuit}_{0,par}([u'])}} |\mathcal{M}^{\heartsuit}_{0,par}([u'])| \Big(w \otimes \mathcal{A}([f], u') \cdot (T + T^{-1})^{V_i^*}\Big)$$

*where $\mathcal{M}^{\heartsuit}_{0,par}([u'])$ is the moduli space of parametric "pseudo-hearts", i.e. pseudo-holomorphic maps whose domain is the degenerate annulus on the right and prescribed Lagrangian boundary conditions. The parameter is given by the relative position of the two boundary marked points as in [Liu20, Fig. 10, top left side].*




As suggested by Fig. (3) for $k = 2$, the section $s$ is decomposed in a sequence of sections $s_1, \ldots, s_k$, each of them contributing to the relevant map $\widetilde{\sigma_i^{Lag}}$. **By iterating the cone construction as in [Kea14, Prop 6.3] for each index $i$ in $CF(L_0, \tau_{V_i} L_1^{i-1}; \mathcal{L}_{\{V_j\}^i})$, we obtain a chain homotopy between the (total) section counting map**

$$\widetilde{\sigma_{tot}^{Lag}} : CF(L_0, L_1) \to CF(L_0, \tau_{V_k} L_1^{k-1}; \mathcal{L}_{\{V_j\}^k}))$$

**and a map into a rather big chain complex generated by the iterated cone which can be made totally explicit.** All the elementary pieces of such map are variations of the ones appearing in our theorem above, and therefore can be computed explicitly in terms of the geometry of the general fiber $F$ and the thimbles.

### 1.3 The fixed point case

We are currently working on the enriched version of the exact triangle for FP Floer homology (2) and what is the correct framework to iterate it. **An easy adaptation of the reasoning done for the Lagrangian case gives a fully geometric proof of the exactness of** (2). As for the other case, the neck-stretching argument from [Ped24] is a crucial ingredient for that.
**Since the Lefschetz fibration defining $\sigma^{\mathbf{Lag}}$ naturally sits inside the one defining $\sigma^{FP}$, local coefficients should be defined in the same way as in the Lagrangian case to ensure compatibility.**

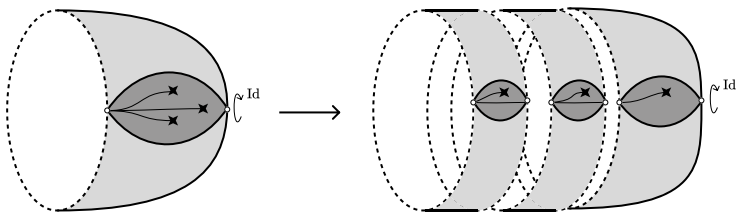



**Figure 5:** The darker disk ($D^{\text{Lag}}$) naturally sits inside the lighter one ($D^{FP}$) and after stretching the latter, we obtain the decomposition for the Lagrangian case.

**The addition map to add the contribution of each section should respect this compatibility too**, "forcing" it to only keep track of the portion of the section over $D^{\text{Lag}}$ in the appropriate relative homology group. This is not necessarily a restriction for our purposes, since the total fibration $X_{D^{FP}} \to D^{FP}$ deformation retracts to $X_{D^{\text{Lag}}} \to D^{\text{Lag}}$

## 2. Future directions

**We believe that our machinery should carry over to the quilted setting** of Wehrheim and Woodward [WW10]. This is a crucial step for two reasons: in order to extend our approach to a general element of $H^2(X;\mathbb{Z})$, we might need to work with the so-called Hilbert relative scheme $X_r(\pi)$ associated to the fibration $\pi$ [DS03]. This construction has the property of transforming a $r$-multisection of $X$ into a section of $X_r(\pi)$, but monodromies there are *fibered* twists rather than honest twists, requiring the quilted package. **With these formulas in hand, a completely explicit description of the SW invariants of a symplectic 4-manifold in terms of the positive factorization of the identity and the geometry of the general fiber should be achievable.** Such formula could give some kind of explicit restriction or conditions on whether a 2nd-degree homology class in $X$ admits a pseudo-holomorphic representative.
**Another consequence of working in this more general context would be that the quilted framework could make the proof of the following conjecture by P. Seidel within reach.**

**Conjecture.** *([Sei01, Conj. 6.1]) Let $\phi$ be the global monodromy of an exact Lefschetz fibration $\pi : X \to D$, then there is a long exact sequence*

$$\begin{tikzcd} HF^*(\phi,+) \arrow[r] & H^*(X;\mathbb{Z}_2) \arrow[r] & HH^*(\mathcal{A},\mathcal{A}) \arrow[l, bend left] \end{tikzcd} . \tag{6}$$

*where $HH^*(\mathcal{A},\mathcal{A})$ is the Hochschild cohomology of the Fukaya-Seidel category of $\pi$, $\mathcal{A}$. $HF^*(\phi,+)$ is a version of FP Floer cohomology for manifolds with boundary.*

To prove that, it might be advantageous to use the Lagrangian Floer interpretation of the Fixed Point Floer cohomology groups as $HF^*(\Delta, (\mathrm{Id}\times\phi)\Delta)$, where $\Delta$ is the diagonal in $F_- \times F$. In this scenario, $(\mathrm{Id}\times\tau_V)$ is a *fibered twist*, making the quilted technology necessary.
My work in [Ped24] also offers potential future research directions: there is an obvious next step in generalizing the isomorphism to composition of Dehn twists along not necessarily disjoint spheres, which could be achieved with a perturbation scheme similar to the one studied in [Eft04], and then using virtual techniques to prove the result for general symplectic manifolds. Another future goal inspired by my previous work is **to explore whether neck-stretching arguments can be applied to study more general symplectomorphisms in higher dimensions**: to this end, the multiple cut/neck-stretching technology developed by Venugopalan and Woodward [VW22] could be an interesting tool to explore. It allows for stretching along certain families of hypersurfaces, giving rise to a decomposition of the original symplectic manifold into pieces in which the Floer homology of such automorphism could be easier to understand. This can lead to some insights into the Fixed point homology of the symplectomorphism on the original manifold.

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